Pointwise Qualitative Factorization of 2-forms into 1-forms over R^4

Kenneth Boyd
and
Franci, Math Consultant™ V0.3.9.6+

Abstract: Assume an arbitrary 2-form over R^4, and a fixed point x in R^4 for which the signs [+,0,-] of all six coordinates of the 2-form are known, are given. If the 2-form factors into a wedge product of two 1-forms, a complete classification of the possible combinations of signs of the coordinates of the 1-forms (at x) is presented. Also, it is shown that some sign combinations in the coordinates of the 2-form preclude the 2-form being factorizable into a wedge product of two 1-forms.

Why would we want this problem solved?

An elementary example is relativistic gyroscopic precession. A spinning body (an electron, the Earth, etc.) that accelerates because forces act on its center of mass precesses its axis of rotation according to the formula

$$d\mathbf{S}/d\tau = (\mathbf{u} \wedge \mathbf{a})\cdot\mathbf{S}$$

where τ is proper time measured along the object's worldline, **S** is the angular-moment 4-vector of the object, **u** is the 4-velocity of the object, and **a** is the 4-acceleration applied to the center of mass of the object. The pointwise algebraic constraints discussed in this paper are constraints on the 2-form , which is known to be a wedge product of two 1-forms. [From another point of view, the angular momentum is Fermi-Walker transported.]

By definition, all 4-vectors, 1-forms, and 2-forms mentioned are over a 4-dimensional space-time manifold, which locally cannot be distinguished from R^4. Thus, the pointwise algebraic constraints, to be discussed, are applicable in any space-time manifold, except at singularities.

Problem definition

Use the standard basis for $\mathbb{R}^4$, and the induced standard basis for 1-forms and 2-forms over $\mathbb{R}^4$. Denote the coordinates (in $\mathbb{R}$) of a fixed 2-form ω (at a fixed point x in $\mathbb{R}^4$), by

$$\omega(x)=c_{12}dx^1\wedge dx^2+c_{13}dx^1\wedge dx^3+c_{14}dx^1\wedge dx^4+c_{23}\,dx^2\wedge dx^3+c_{24}dx^2\wedge dx^4+c_{34}dx^3\wedge dx^4$$

Also, (assuming that 1-forms α, β exist such that $\omega = \alpha\wedge\beta$), denote these in coordinates (in $\mathbb{R}$) by

$$\alpha(x)=a_1dx^1+a_2dx^2+a_3dx^3+a_4dx^4$$
$$\beta(x)=b_1dx^1+b_2dx^2+b_3dx^3+b_4dx^4$$

Recall that in (assumed possible) the coordinate computation of $\omega = \alpha\wedge\beta$,

$$c_{12}=a_1b_2-a_2b_1 \quad c_{13}=a_1b_3-a_3b_1 \quad c_{14}=a_1b_4-a_4b_1$$
$$c_{34}=a_3b_4-a_4b_3 \quad c_{24}=a_2b_4-a_4b_2 \quad c_{23}=a_2b_3-a_3b_2$$

We sometimes use i,j = 1..4. [i<j] when used as subscripts in c_{ij}; otherwise, i and j are not necessarily related. Note that in the above translation, using the permutation [ab] on the above equations is the coordinate version of writing $-\omega = \beta\wedge\alpha$. We will use this freely when using the computational lemmas [i.e.: the permutation [ab] may be applied globally in a lemma, provided all of the c_{ij} are negated.]

Finally, we require that it be known, for each of the 2-form coordinates c_{ij}, whether it is positive, zero, or negative [$c_{ij}>0$, $c_{ij}=0$, $c_{ij}<0$]. Parts of the classification will consider two of these cases at once, for various c_{ij}.

From this information, we want to know:

- Is it possible to rule out a factorization $\omega = \alpha\wedge\beta$, just from the sign information for the c_{ij}?
- If we have failed to rule out such a factorization $\omega = \alpha\wedge\beta$, what does the sign information for the c_{ij} allow us to infer about sign information for the a_i, b_j?

Computing the action of the symmetric group S_4 on the coordinates

It will be useful, in reducing the amount of work in the classification, to have an explicit description of the action of the symmetric group S_4 on the 2-form coordinates c_{ij}, as well as the effects of this action on the constraints. [For the rest of this paper, S_4 shall always refer to the symmetric group S_4.]

The overall results are as follows:

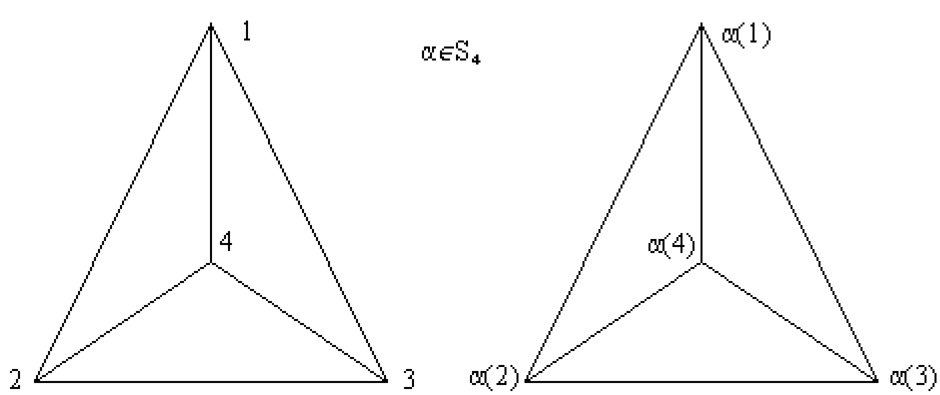
id	[12]	[13]	[14]	[23]	[24]	[34]	[123]	[132]	[124]	[142]	[134]
c_{12}	$-c_{12}$	$-c_{23}$	$-c_{24}$	c_{13}	c_{14}	c_{12}	c_{23}	$-c_{13}$	c_{24}	$-c_{14}$	$-c_{23}$
c_{13}	c_{23}	$-c_{13}$	$-c_{34}$	c_{12}	c_{13}	c_{14}	$-c_{12}$	$-c_{23}$	c_{23}	$-c_{34}$	c_{34}
c_{14}	c_{24}	c_{34}	$-c_{14}$	c_{14}	c_{12}	c_{13}	c_{24}	c_{34}	$-c_{12}$	$-c_{24}$	$-c_{13}$
c_{23}	c_{13}	$-c_{12}$	c_{23}	$-c_{23}$	$-c_{34}$	c_{24}	$-c_{13}$	c_{12}	$-c_{34}$	c_{14}	c_{24}
c_{24}	c_{14}	c_{24}	$-c_{12}$	c_{34}	$-c_{24}$	c_{23}	c_{34}	c_{14}	$-c_{14}$	c_{12}	$-c_{12}$
c_{34}	c_{34}	c_{14}	$-c_{13}$	c_{24}	$-c_{23}$	$-c_{34}$	c_{14}	c_{24}	$-c_{13}$	$-c_{23}$	$-c_{14}$

[143]	[234]	[243]	[12] [34]	[13] [24]	[14] [23]	[1234]	[1243]	[1324]	[1342]	[1423]	[1432]
$-c_{24}$	c_{13}	c_{14}	$-c_{12}$	c_{34}	$-c_{34}$	c_{23}	c_{24}	c_{34}	$-c_{13}$	$-c_{34}$	$-c_{14}$
$-c_{14}$	c_{14}	c_{12}	c_{24}	$-c_{13}$	$-c_{24}$	c_{24}	$-c_{12}$	$-c_{23}$	c_{34}	$-c_{14}$	$-c_{24}$
$-c_{34}$	c_{12}	c_{13}	c_{23}	$-c_{23}$	$-c_{14}$	$-c_{12}$	c_{23}	$-c_{13}$	$-c_{23}$	$-c_{24}$	c_{34}
$-c_{12}$	c_{34}	$-c_{24}$	c_{14}	$-c_{14}$	$-c_{23}$	c_{34}	$-c_{14}$	$-c_{24}$	c_{14}	$-c_{13}$	c_{12}
c_{23}	$-c_{23}$	$-c_{34}$	c_{13}	$-c_{24}$	$-c_{13}$	$-c_{13}$	$-c_{34}$	$-c_{14}$	c_{12}	$-c_{23}$	c_{13}
c_{13}	$-c_{24}$	c_{23}	$-c_{34}$	c_{12}	$-c_{12}$	c_{14}	c_{13}	$-c_{12}$	$-c_{24}$	c_{12}	c_{23}

A graphical method of computing the above action is illustrated as follows:

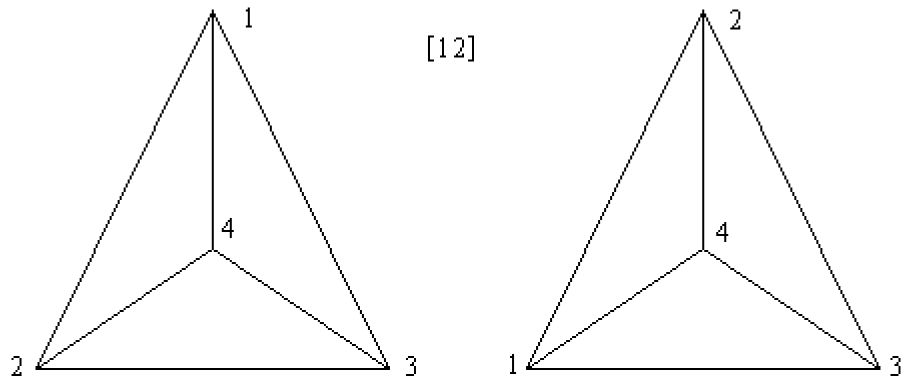
The edges with vertices i,j index the coefficient c_{ij}, in standard orientation. It will be convenient to define [for $1 \le i < j \le 4$]

$$c_{ji} := -c_{ij}$$



Then we can write that $\alpha \varepsilon S_4$ maps c_{ij} to $c_{\alpha(i)\alpha(j)}$.

As an example, consider the image of the transposition [12]:



Either the graphical method, or the algebraic-table method, may be used in calculating the equivalence classes of the various cases in the classification.

Computational facts

Besides the sign information about a_ib_j relative to a_i and b_j, and c_{ij} relative to a_ib_j-a_jb_i,

	$b_j<0$	$b_j=0$	$b_j>0$			$a_jb_i<0$	$a_jb_i=0$	$a_jb_i>0$
$a_i<0$	$a_ib_j>0$	$a_ib_j=0$	$a_ib_j<0$		$a_ib_j<0$	$c_{ij}?0$	$c_{ij}<0$	$c_{ij}<0$
$a_i=0$	$a_ib_j=0$	$a_ib_j=0$	$a_ib_j=0$		$a_ib_j=0$	$c_{ij}>0$	$c_{ij}=0$	$c_{ij}<0$
$a_i>0$	$a_ib_j<0$	$a_ib_j=0$	$a_ib_j>0$		$a_ib_j>0$	$c_{ij}>0$	$c_{ij}>0$	$c_{ij}?0$

Note: $c_{ij}<0$ implies [$a_i<0$ AND $b_j>0$]
OR [$a_i>0$ AND $b_j<0$]
OR [$a_j<0$ AND $b_i<0$]
OR [$a_j>0$ AND $b_i>0$]

Note: $c_{ij}>0$ implies [$a_i<0$ AND $b_j<0$]
OR [$a_i>0$ AND $b_j>0$]
OR [$a_j<0$ AND $b_i>0$]
OR [$a_j>0$ AND $b_i<0$]

Note: c_{ij}=0 implies [a_i<0 AND a_j<0 AND b_i<0 AND b_j<0]
OR [a_i>0 AND a_j<0 AND b_i<0 AND b_j>0]
OR [a_i<0 AND a_j>0 AND b_i>0 AND b_j<0]
OR [a_i>0 AND a_j>0 AND b_i>0 AND b_j>0]
OR [a_i=0 AND a_j=0]
OR [a_i=0 AND b_i=0]
OR [b_j=0 AND a_j=0]
OR [b_j=0 AND b_i=0]
OR [a_i<0 AND a_j<0 AND b_i>0 AND b_j>0]
OR [a_i>0 AND a_j<0 AND b_i>0 AND b_j<0]
OR [a_i<0 AND a_j>0 AND b_i<0 AND b_j>0]
OR [a_i>0 AND a_j>0 AND b_i<0 AND b_j<0]

Note [there is an analog for $a_i \neq 0 \neq a_j$]:

$$c_{ij}=a_i b_j - a_j b_i > 0 \, \textit{iff} \, \frac{a_i}{b_i} > \frac{a_j}{b_j}$$

$$b_i \neq 0 \neq b_j \, \textit{IMPLIES} [c_{ij}=a_i b_j - a_j b_i = 0 \, \textit{iff} \, \frac{a_i}{b_i} = \frac{a_j}{b_j}]$$

$$c_{ij}=a_i b_j - a_j b_i < 0 \, \textit{iff} \, \frac{a_i}{b_i} < \frac{a_j}{b_j}$$

Note: the configuration $c_{ij}=0=c_{ik}$, $c_{jk}\neq 0$, implies that at least one of a_i, a_j, a_k is zero, and also that at least one of b_i, b_j, and b_k is zero. [Otherwise, the calculational note about comparing quotients applies, and $c_{jk}=0$ — contradiction.]. As a shorthand, we refer to instances of the above comment as ALO1(i,j,k); the arguments do not commute, and j<k by convention.

In fact, the conditions yielding ALO1(i,j,k) actually imply that [$a_i=0=b_i$]. [This will be Lemma 2, later.] We will prove this directly at Lemma 2. A way to visualize this is that the cross-product of two vectors in $\mathbb{R}^3$ is the dual of the wedge product of the vectors (reinterpreted coordinate-wise as 1-forms) over $\mathbb{R}^3$. The hypothesis then translates to requiring the cross-product to be a vector that is non-zero only in the first coordinate; since the cross-product, geometrically, is perpendicular to both of the vectors it was computed from, both of the vectors it was computed from have first coordinate zero. [Actually, we will (re)prove that the visualization is a correct interpretation of the cross-product over $\mathbb{R}^3$.]

Computational Lemmas

Lemma 0

$$[[a_i=0 \text{ OR } b_j=0] \text{ AND } c_{ij}<0] \text{ IMPLIES } a_jb_i>0$$
$$[[a_i=0 \text{ OR } b_j=0] \text{ AND } c_{ij}=0] \text{ IMPLIES } a_jb_i=0$$
$$[[a_i=0 \text{ OR } b_j=0] \text{ AND } c_{ij}>0] \text{ IMPLIES } a_jb_i<0$$
$$[[a_j=0 \text{ OR } b_i=0] \text{ AND } c_{ij}<0] \text{ IMPLIES } a_ib_j<0$$
$$[[a_j=0 \text{ OR } b_i=0] \text{ AND } c_{ij}=0] \text{ IMPLIES } a_ib_j=0$$
$$[[a_j=0 \text{ OR } b_i=0] \text{ AND } c_{ij}>0] \text{ IMPLIES } a_ib_j>0$$

Sketch of calculational proof: element chase of the first two tables in the computational facts. The first three use the hypothesis to single out the second row of the RHS table, while the last three use the hypothesis to single out the second column of the RHS table.

Lemma 1

$$[[a_i=0 \text{ OR } [b_j=0 \text{ AND } b_k=0]] \text{ AND } c_{ij}\neq 0 \text{ AND } c_{ik}=0] \text{ IMPLIES } a_k=0$$

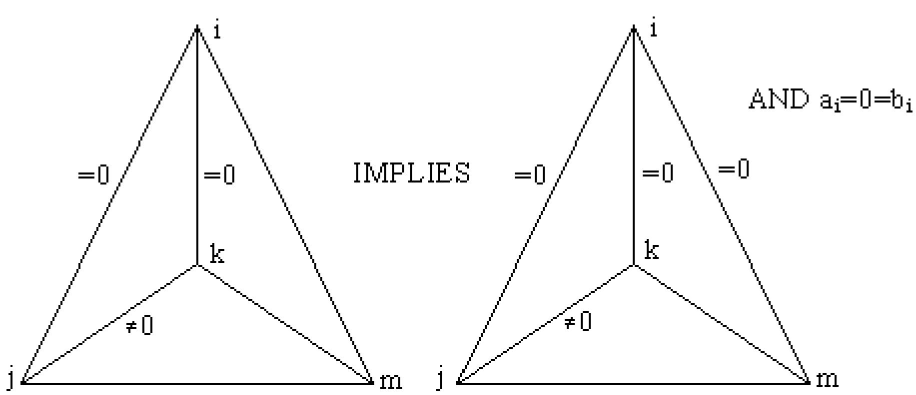
Sketch of calculational proof:

Recall that $c_{ij}=a_ib_j - a_jb_i$. Use Lemma 0 with $c_{ij}\neq 0$ to get $a_jb_i\neq 0$ [making $b_i\neq 0$], and with $c_{ik}=0$ to get $a_kb_i=0$. Since $b_i\neq 0$, and R has no zero divisors, $a_k=0$.

[**NOTE**: other variations on the above hypotheses [two 2-form coordinates with a common index, and a variable entering into at least one of the coordinate calculations, are constrained] lead to repeats of instances of lemma 0. I have suppressed the repeats of lemma 0 in lemma 1.]

Lemma 2:

$$[c_{ij}=0=c_{ik} \text{ AND } c_{jk}\neq 0] \text{ IMPLIES } [a_i=0=b_i \text{ AND } c_{im}=0]$$



Sketch of calculational proof:

We have (using calculational notes about the hypothesized entries):

[[$a_i<0$ AND $a_j<0$ AND $b_i<0$ AND $b_j<0$]
OR [$a_i>0$ AND $a_j<0$ AND $b_i<0$ AND $b_j>0$]
OR [$a_i<0$ AND $a_j>0$ AND $b_i>0$ AND $b_j<0$]
OR [$a_i>0$ AND $a_j>0$ AND $b_i>0$ AND $b_j>0$]
OR [$a_i=0$ AND $a_j=0$]
OR [$a_i=0$ AND $b_i=0$]
OR [$b_j=0$ AND $a_j=0$]
OR [$b_j=0$ AND $b_i=0$]
OR [$a_i<0$ AND $a_j<0$ AND $b_i>0$ AND $b_j>0$]
OR [$a_i>0$ AND $a_j<0$ AND $b_i>0$ AND $b_j<0$]
OR [$a_i<0$ AND $a_j>0$ AND $b_i<0$ AND $b_j>0$]
OR [$a_i>0$ AND $a_j>0$ AND $b_i<0$ AND $b_j<0$]]
AND [[$a_i<0$ AND $a_k<0$ AND $b_i<0$ AND $b_k<0$]
OR [$a_i>0$ AND $a_k<0$ AND $b_i<0$ AND $b_k>0$]
OR [$a_i<0$ AND $a_k>0$ AND $b_i>0$ AND $b_k<0$]
OR [$a_i>0$ AND $a_k>0$ AND $b_i>0$ AND $b_k>0$]
OR [$a_i=0$ AND $a_k=0$]
OR [$a_i=0$ AND $b_i=0$]
OR [$b_k=0$ AND $a_k=0$]
OR [$b_k=0$ AND $b_i=0$]
OR [$a_i<0$ AND $a_k<0$ AND $b_i>0$ AND $b_k>0$]
OR [$a_i>0$ AND $a_k<0$ AND $b_i>0$ AND $b_k<0$]
OR [$a_i<0$ AND $a_k>0$ AND $b_i<0$ AND $b_k>0$]
OR [$a_i>0$ AND $a_k>0$ AND $b_i<0$ AND $b_k<0$]]
AND [[$a_j\neq0$ AND $b_k\neq0$]
OR [$a_k\neq0$ AND $b_j\neq0$]]

The third OR-clause implies (but not conversely) that [$a_j \neq 0$ OR $b_j \neq 0$] and [$a_k \neq 0$ OR $b_k \neq 0$]. Applying this to the first and second OR-clauses yields:

[[$a_i<0$ AND $a_j<0$ AND $b_i<0$ AND $b_j<0$]
OR [$a_i>0$ AND $a_j<0$ AND $b_i<0$ AND $b_j>0$]
OR [$a_i<0$ AND $a_j>0$ AND $b_i>0$ AND $b_j<0$]
OR [$a_i>0$ AND $a_j>0$ AND $b_i>0$ AND $b_j>0$]
OR [$a_i=0$ AND $a_j=0$]
OR [$a_i=0$ AND $b_i=0$]
OR [$b_j=0$ AND $b_i=0$]
OR [$a_i<0$ AND $a_j<0$ AND $b_i>0$ AND $b_j>0$]
OR [$a_i>0$ AND $a_j<0$ AND $b_i>0$ AND $b_j<0$]
OR [$a_i<0$ AND $a_j>0$ AND $b_i<0$ AND $b_j>0$]
OR [$a_i>0$ AND $a_j>0$ AND $b_i<0$ AND $b_j<0$]]
AND [[$a_i<0$ AND $a_k<0$ AND $b_i<0$ AND $b_k<0$]
OR [$a_i>0$ AND $a_k<0$ AND $b_i<0$ AND $b_k>0$]
OR [$a_i<0$ AND $a_k>0$ AND $b_i>0$ AND $b_k<0$]
OR [$a_i>0$ AND $a_k>0$ AND $b_i>0$ AND $b_k>0$]
OR [$a_i=0$ AND $a_k=0$]
OR [$a_i=0$ AND $b_i=0$]
OR [$b_k=0$ AND $b_i=0$]
OR [$a_i<0$ AND $a_k<0$ AND $b_i>0$ AND $b_k>0$]
OR [$a_i>0$ AND $a_k<0$ AND $b_i>0$ AND $b_k<0$]
OR [$a_i<0$ AND $a_k>0$ AND $b_i<0$ AND $b_k>0$]
OR [$a_i>0$ AND $a_k>0$ AND $b_i<0$ AND $b_k<0$]]
AND [[$a_j \neq 0$ AND $b_k \neq 0$]
OR [$a_k \neq 0$ AND $b_j \neq 0$]]

Now, ALO1(i,j,k) implies that when we expand the AND of the first two OR-clauses: the first four, and the last four, of the terms in the first OR-clause cannot successfully expand with any of the eleven terms in the second OR-clause. [The first four and the last four terms of the second OR-clause violate ALO1(i,j,k), and the central three terms of the second OR-clause directly contradict the term from the first OR-clause that we are expanding.] Similar reasoning holds with the first four, and the last four, terms of the second OR-clause with respect to the first OR-clause. This allows reducing the above to:

[[$a_i=0$ AND $a_j=0$]
OR [$a_i=0$ AND $b_i=0$]
OR [$b_j=0$ AND $b_i=0$]]
AND [[$a_i=0$ AND $a_k=0$]
OR [$a_i=0$ AND $b_i=0$]
OR [$b_k=0$ AND $b_i=0$]]
AND [[$a_j \neq 0$ AND $b_k \neq 0$]
OR [$a_k \neq 0$ AND $b_j \neq 0$]]

Now, expand the AND of the first two OR-clauses:

[[[a_i=0 AND a_j=0] AND [a_i=0 AND a_k=0]]

OR [[a_i=0 AND a_j=0] AND [a_i=0 AND b_i=0]]

OR [[a_i=0 AND a_j=0] AND [b_k=0 AND b_i=0]]

OR [[a_i=0 AND b_i=0] AND [a_i=0 AND a_k=0]]

OR [[a_i=0 AND b_i=0] AND [a_i=0 AND b_i=0]]

OR [[a_i=0 AND b_i=0] AND [b_k=0 AND b_i=0]]

OR [[b_j=0 AND b_i=0] AND [a_i=0 AND a_k=0]]

OR [[b_j=0 AND b_i=0] AND [a_i=0 AND b_i=0]]

OR [[b_j=0 AND b_i=0] AND [b_k=0 AND b_i=0]]]

AND [[$a_j \neq 0$ AND $b_k \neq 0$]

OR [$a_k \neq 0$ AND $b_j \neq 0$]]

i.e. (Redundancy elimination only, no sorting, remove second duplicate term: first OR-clause)

[[a_i=0 AND a_j=0 AND a_k=0]

OR [a_i=0 AND a_j=0 AND b_i=0]

OR [a_i=0 AND a_j=0 AND b_k=0 AND b_i=0]

OR [a_i=0 AND b_i=0 AND a_k=0]

OR [a_i=0 AND b_i=0]

OR [a_i=0 AND b_i=0 AND b_k=0]

OR [b_j=0 AND b_i=0 AND a_i=0 AND a_k=0]

OR [b_j=0 AND b_i=0 AND a_i=0]

OR [b_j=0 AND b_i=0 AND b_k=0]]

AND [[$a_j \neq 0$ AND $b_k \neq 0$]

OR [$a_k \neq 0$ AND $b_j \neq 0$]]

Now, the second OR-clause also yields [$a_j \neq 0$ OR $a_k \neq 0$] and [$b_j \neq 0$ OR $b_k \neq 0$]. This removes the first and last clauses of the first OR-clause:

[[a_i=0 AND a_j=0 AND b_i=0]

OR [a_i=0 AND a_j=0 AND b_k=0 AND b_i=0]

OR [a_i=0 AND b_i=0 AND a_k=0]

OR [a_i=0 AND b_i=0]

OR [a_i=0 AND b_i=0 AND b_k=0]

OR [b_j=0 AND b_i=0 AND a_i=0 AND a_k=0]

OR [b_j=0 AND b_i=0 AND a_i=0]]

AND [[$a_j \neq 0$ AND $b_k \neq 0$]

OR [$a_k \neq 0$ AND $b_j \neq 0$]]

We can now factor [a_i=0 AND b_i=0] out of the first OR-clause:

[a_i=0 AND b_i=0]
AND [[a_j=0]
OR [a_j=0 AND b_k=0]
OR [a_k=0]
OR [true]
OR [b_k=0]
OR [b_j=0 AND a_k=0]
OR [b_j=0]]
AND [[$a_j \neq 0$ AND $b_k \neq 0$]
OR [$a_k \neq 0$ AND $b_j \neq 0$]]

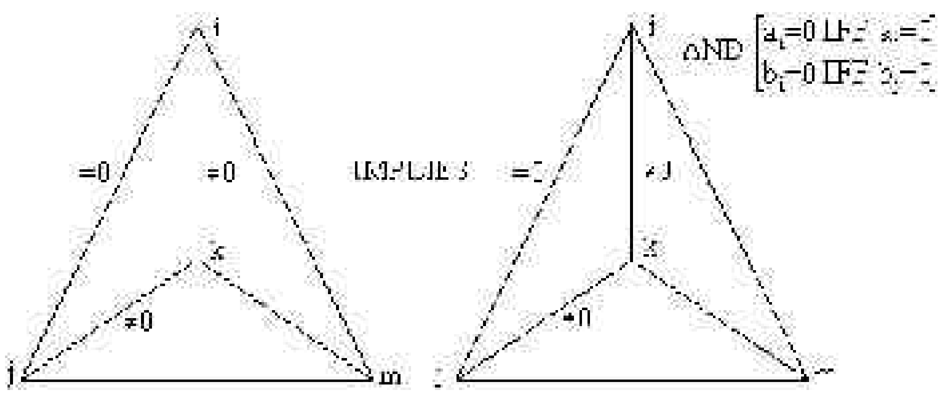
i.e. [the first OR-clause is trivially true]

[a_i=0 AND b_i=0]
AND [[$a_j \neq 0$ AND $b_k \neq 0$]
OR [$a_k \neq 0$ AND $b_j \neq 0$]]

It is now trivial that c_{im}=0.

Lemma 3:

[c_{ij}=0 AND $c_{ik} \neq 0$ AND $c_{jk} \neq 0$] IMPLIES [a_i=0 IFF a_j=0]
swap a, b
[c_{ij}=0 AND $c_{ik} \neq 0$ AND $c_{jk} \neq 0$] IMPLIES [b_i=0 IFF b_j=0]



Sketch of calculational proof:

Assume c_{ij}=0, $c_{ik} \neq 0 \neq c_{jk}$. Then Lemma 1 yields both 'a_i=0 implies a_j=0' [$c_{ik} \neq 0$, c_{ij}=0] and 'a_j=0 implies a_i=0' [$c_{jk} \neq 0$, c_{ij}=0], i.e. 'a_i=0 iff a_j=0'.

Lemma 4:

$$c_{13}c_{24} = c_{12}c_{34}+c_{14}c_{23}$$

Sketch of calculational proof:

$$c_{12}c_{34}+c_{14}c_{23} = [a_1b_2-a_2b_1][a_3b_4-a_4b_3]+[a_1b_4-a_4b_1][a_2b_3-a_3b_2]$$
$$= a_1b_2a_3b_4 - a_1b_2a_4b_3 - a_2b_1a_3b_4 + a_2b_1a_4b_3 + a_1b_4a_2b_3 - a_1b_4a_3b_2 - a_4b_1a_2b_3 + a_4b_1a_3b_2$$
$$= - a_1b_2a_4b_3 - a_2b_1a_3b_4 + a_1b_4a_2b_3 + a_4b_1a_3b_2 = [a_1b_3 - a_3b_1][a_2b_4 - a_4b_2] = c_{13}c_{24}$$

It is a routine check that this equation is fixed under the S_4 vertex action. [The simplest way is to verify that the sign of all three terms is negated under the transpositions [14], [24], [34] — which are a generating set for S_4.]

The classification

We will first do a preliminary classification, based on whether $c_{ij} = 0$, or $c_{ij} \neq 0$. We first enumerate the distinct cases (for the c_{ij}) under the action of S_4 on the indexes 1..4, and then explicitly describe the possible cases for the a_i and b_j, for each case of the c_{ij}.

We first enumerate representatives of the equivalence classes of the constraints:

Partial case	c_{12}	c_{13}	c_{14}	c_{23}	c_{24}	c_{34}	Images under S_4 action	Total cases per image	Total cases indexed by partial case
0	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	1	$2^6=64$	64
1	=0	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	6	$2^5=32$	192
2A	=0	=0	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	12	$2^4=16$	192
2B	=0	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	=0	3	$2^4=16$	48
3A	=0	=0	=0	$\neq 0$	$\neq 0$	$\neq 0$	4	$2^3=8$	32
3B	=0	$\neq 0$	$\neq 0$	=0	$\neq 0$	=0	12	$2^3=8$	96
3C	=0	=0	$\neq 0$	=0	$\neq 0$	$\neq 0$	4	$2^3=8$	32
4A	=0	$\neq 0$	=0	=0	$\neq 0$	=0	3	$2^2=4$	12
4B	=0	=0	=0	=0	$\neq 0$	$\neq 0$	12	$2^2=4$	48
5	=0	=0	=0	=0	=0	$\neq 0$	6	$2^1=2$	12
6	=0	=0	=0	=0	=0	=0	1	$2^0=1$	1

The last column is for how many cases in the total classification are indexed by the above partial cases. The left-hand factor is the number of images of the representative under the S_4 action on the vertices; the right-hand factor is 2 to the number of $\neq 0$ constraints in the partial case.
As a cross-check, note that the sum of the counts is 729[$= 3^6$], the number of cases to be dealt with in the total classification. Also, note that all of these cases are preserved by commuting the 1-forms in the wedge product [coordinate-wise, swapping a and b with the permutation [ab].] We

will use this symmetry freely, without comment, in the preliminary classification.

Note: Cases 2A, 3B, and 4A directly violate both Lemma 2 and Lemma 4, thus cannot factor into a wedge product of 1-forms. We only need to analyze the other cases.

Case 0: [$c_{12}\neq0$, $c_{13}\neq0$, $c_{14}\neq0$, $c_{23}\neq0$, $c_{24}\neq0$, $c_{34}\neq0$]
[Symmetry on vertices: S_4]

In Case 0, from all 6 $c_{ij}\neq0$ we have:

[$a_1\neq0$ AND $b_2\neq0$] OR [$a_2\neq0$ AND $b_1\neq0$]
[$a_1\neq0$ AND $b_3\neq0$] OR [$a_3\neq0$ AND $b_1\neq0$]
[$a_1\neq0$ AND $b_4\neq0$] OR [$a_4\neq0$ AND $b_1\neq0$]
[$a_2\neq0$ AND $b_3\neq0$] OR [$a_3\neq0$ AND $b_2\neq0$]
[$a_2\neq0$ AND $b_4\neq0$] OR [$a_4\neq0$ AND $b_2\neq0$]
[$a_3\neq0$ AND $b_4\neq0$] OR [$a_4\neq0$ AND $b_3\neq0$]

Lemma 4 rules out the following 16 subcases (either verify the S_4 vertex action and the kernel, or all 16 cases):

```
Computation of images under vertex S4 action of
c_12<0, c_13<0, c_14<0, c_23<0, c_24>0, c_34<0
            12 13 14 23 24 34
[id]        <  <  <  <  >  <  [234],[243]       // kernel
[23]        <  <  <  >  <  >  [24],[34]
[14]        <  >  >  <  >  >  [1423],[1432]
[13][24]    <  >  >  >  <  <  [132],[134]
[12]        >  <  >  <  <  <  [1243],[1234]
[14][23]    >  <  >  >  >  >  [142],[143]
[12][34]    >  >  <  <  <  >  [123],[124]
[13]        >  >  <  >  >  <  [1324],[1342]

This does not contain its sign-flip, qualitatively.
Thus, the commuted case is depicted by:
Dual computation of images under vertex S4 action of
c_12>0, c_13>0, c_14>0, c_23>0, c_24<0, c_34>0
            12 13 14 23 24 34
[id]        >  >  >  >  <  >  [234],[243]       // kernel
[23]        >  >  >  <  >  <  [24],[34]
[14]        >  <  <  >  <  <  [1423],[1432]
[13][24]    >  <  <  <  >  >  [132],[134]
[12]        <  >  <  >  >  >  [1243],[1234]
[14][23]    <  >  <  <  <  <  [142],[143]
[12][34]    <  <  >  >  >  <  [123],[124]
[13]        <  <  >  <  <  >  [1324],[1342]
```

Case 1: [$c_{12}=0$, $c_{13}\neq0$, $c_{14}\neq0$, $c_{23}\neq0$, $c_{24}\neq0$, $c_{34}\neq0$]
[Symmetry on vertices: <[12],[34]> subgroup S_4]

Lemma 3 yields [$c_{12}=0$, $c_{13}\neq 0$, $c_{14}\neq 0$]:

$$a_1=0 \text{ IFF } a_2=0$$
$$b_1=0 \text{ IFF } b_2=0$$

From $c_{13}\neq 0$, $c_{14}\neq 0$, $c_{23}\neq 0$, $c_{24}\neq 0$, $c_{34}\neq 0$, we have (after using above):

$$[a_1\neq 0 \text{ AND } b_3\neq 0] \text{ OR } [a_3\neq 0 \text{ AND } b_1\neq 0]$$
$$[a_1\neq 0 \text{ AND } b_4\neq 0] \text{ OR } [a_4\neq 0 \text{ AND } b_1\neq 0]$$
$$[a_3\neq 0 \text{ AND } b_4\neq 0] \text{ OR } [a_4\neq 0 \text{ AND } b_3\neq 0]$$

Lemma 4 rules out the following subcases (again, either verify the S_4 action and the kernels, or all of the cases):

```
      12    13    14    23    24    34
...
Type II: failed Lemma 4
      =     <     <     <     >     <     [id]        // kernel
      =     <     >     <     <     <     [12]
      =     <     <     >     <     >     [34]
      =     >     <     <     <     >     [12][34]
Type III: failed Lemma 4
      =     <     <     <     >     >     [id]        // kernel
      =     <     >     <     <     >     [12]
      =     <     <     >     <     <     [34]
      =     >     <     <     <     <     [12][34]
...
Type III sign-flip: failed Lemma 4
      =     >     >     >     <     <     [id]        // kernel
      =     >     <     >     >     <     [12]
      =     >     >     <     >     >     [34]
      =     <     >     >     >     >     [12][34]
Type II sign-flip: failed Lemma 4
      =     >     >     >     <     >     [id]        // kernel
      =     >     <     >     >     >     [12]
      =     >     >     <     >     <     [34]
      =     <     >     >     >     <     [12][34]
```

Case 2B: [$c_{12}=0$, $c_{13}\neq 0$, $c_{14}\neq 0$, $c_{23}\neq 0$, $c_{24}\neq 0$, $c_{34}=0$]
[Symmetry on vertices: <[12],[34],[13][24]> subgroup S_4]

Lemma 3 yields [$c_{12}=0$, $c_{13}\neq 0$, $c_{14}\neq 0$, $c_{23}\neq 0$, $c_{24}\neq 0$, $c_{34}=0$]:

$$a_1=0 \text{ IFF } a_2=0$$
$$b_1=0 \text{ IFF } b_2=0$$
$$a_3=0 \text{ IFF } a_4=0$$
$$b_3=0 \text{ IFF } b_4=0$$

From $c_{13}\neq 0$, we have:

$$[a_1\neq 0 \text{ AND } b_3\neq 0] \text{ OR } [a_3\neq 0 \text{ AND } b_1\neq 0]$$

Lemma 4 rules out the following subcases (again, either verify the S_4 action and the kernels, or all of the cases):

```
      12    13    14    23    24    34
...
Type II: failed Lemma 4
      =     <     <     <     >     =     [id]        // kernel
      =     <     >     <     <     =     [12]
      =     <     <     >     <     =     [34]
      =     >     <     <     <     =     [12][34]
...
Type II sign-flip: failed Lemma 4
      =     >     >     >     <     =     [id]        // kernel
      =     >     <     >     >     =     [12]
      =     >     >     <     >     =     [34]
      =     <     >     >     >     =     [12][34]
```

Case 3A: [$c_{12}=0$, $c_{13}=0$, $c_{14}=0$, $c_{23}\neq0$, $c_{24}\neq0$, $c_{34}\neq0$]
[Symmetry on vertices: Perm({2,3,4}) subgroup S_4]

Lemma 2 [$c_{12}=0$, $c_{13}=0$, $c_{23}\neq0$] yields for Case 3A:

$$a_1=0=b_1$$

We also have, from $c_{23}\neq0$, $c_{24}\neq0$, $c_{34}\neq0$:

$$[a_2\neq0 \text{ AND } b_3\neq0] \text{ OR } [a_3\neq0 \text{ AND } b_2\neq0]$$
$$[a_2\neq0 \text{ AND } b_4\neq0] \text{ OR } [a_4\neq0 \text{ AND } b_2\neq0]$$
$$[a_3\neq0 \text{ AND } b_4\neq0] \text{ OR } [a_4\neq0 \text{ AND } b_3\neq0]$$

Case 3C: [$c_{12}=0$, $c_{13}=0$, $c_{14}\neq0$, $c_{23}=0$, $c_{24}\neq0$, $c_{34}\neq0$]
[Symmetry on vertices: S_3 subgroup S_4]

Lemma 3 yields [$c_{12}=0$, $c_{14}\neq0$, $c_{24}\neq0$; $c_{13}=0$, $c_{14}\neq0$, $c_{34}\neq0$; $c_{23}=0$, $c_{24}\neq0$, $c_{34}\neq0$]:

$$a_1=0 \text{ IFF } a_2=0 \text{ IFF } a_3=0$$
$$b_1=0 \text{ IFF } b_2=0 \text{ IFF } b_3=0$$

From $c_{14}\neq0$, we also have:

$$[a_1\neq0 \text{ AND } b_4\neq0] \text{ OR } [a_4\neq0 \text{ AND } b_1\neq0]$$

Case 4B: [$c_{12}=0$, $c_{13}=0$, $c_{14}=0$, $c_{23}=0$, $c_{24}\neq0$, $c_{34}\neq0$]
[Symmetry on vertices: <[23]> subgroup S_4]

Lemma 2 yields [$c_{12}=0$, $c_{14}=0$, $c_{24}\neq0$]:

$$a_1=0=b_1$$

Lemma 3 yields [$c_{23}=0$, $c_{24}\neq0$, $c_{34}\neq0$]:

$a_2=0$ IFF $a_3=0$
$b_2=0$ IFF $b_3=0$

Also, from $c_{24}\neq 0$, we have:

$[a_2\neq 0$ AND $b_4\neq 0]$ OR $[a_4\neq 0$ AND $b_2\neq 0]$

Case 5: $[c_{12}=0, c_{13}=0, c_{14}=0, c_{23}=0, c_{24}=0, c_{34}\neq 0]$
[Symmetry on vertices: <[12],[34]> subgroup S_4]

Lemma 2 yields $[c_{13}=0, c_{14}=0, c_{34}\neq 0; c_{23}=0, c_{24}=0, c_{34}\neq 0]$:

$a_1=0=b_1$
$a_2=0=b_2$

Also, from $c_{34}\neq 0$, we have:

$[a_3\neq 0$ AND $b_4\neq 0]$ OR $[a_4\neq 0$ AND $b_3\neq 0]$

Case 6: $[c_{12}=0, c_{13}=0, c_{14}=0, c_{23}=0, c_{24}=0, c_{34}=0]$
[Symmetry on vertices: S_4]

We have, for Case 6:

$\alpha=0$ OR $\beta=0$
OR all of these:
$a_1=0$ IFF $b_1=0$
$a_2=0$ IFF $b_2=0$
$a_3=0$ IFF $b_3=0$
$a_4=0$ IFF $b_4=0$

Sketch of calculational proof:

Since $0\wedge\beta=0=\alpha\wedge 0$, we need only consider the case when $\alpha\neq 0\neq\beta$. In this case, by the problem symmetry we need only to prove $a_1=0$ IFF $b_1=0$.

Suppose $a_1=0$. Then, by definition of c_{ij} [j=2..4], $a_jb_1=0$. Since $\alpha\neq 0$, one of $a_j\neq 0$, yielding $b_1=0$. This proves $a_1=0$ IMPLIES $b_1=0$; swap a and b to get parallel reasoning for the other direction (yielding $a_1=0$ IFF $b_1=0$).

Editors

Critical editing was kindly provided by Professors Robert Burckel and David Yetter at Kansas State University.